Stochastic modeling of mesoscale eddies in oceanic dynamics

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Workshop on Frontiers of Uncertainty Quantification in Fluid Dynamics
Pisa(Italy), September 12, 2019

Motivations

- Better small–scale representation : have explicit formulations and interpretations (*v.s. parametrization with ad hoc tuning*)
- Physical consistency : respect a set of conservation laws (*e.g.* energy, circulation) (*v.s. arbitrary Gaussian forcing*)
- Useful in uncertainty quantification (UQ) : provide more reliable ensemble forecasts (EF) and more effiecient spread for ensemble data assimilation (*v.s. perturbations of initial condition (PIC)*)

Outline

Location Uncertainty (LU) Principles

- Stochastic flow :

$$\mathrm{d}\boldsymbol{X}_t = \underbrace{\boldsymbol{w}(\boldsymbol{X}_t, t)\mathrm{d}t}_{\text{large–scale / resolved}} + \underbrace{\boldsymbol{\sigma}(\boldsymbol{X}_t, t)\mathrm{d}\boldsymbol{B}_t}_{\text{small–scale / unresolved}}$$

- Functional processs :

$$\boldsymbol{\sigma}(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{B}_t = \int_{\Omega} \breve{\boldsymbol{\sigma}}(\boldsymbol{x}, \boldsymbol{y}, t)\mathrm{d}\boldsymbol{B}_t(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$$

$\boldsymbol{B}_t$ is a cyclindrical Wiener process of infinite dimension (in some Hilbert space) and $\breve{\boldsymbol{\sigma}}$ is deterministic symmetric kernel

Location Uncertainty (LU) Principles

- Covariance operator :

$$Q(\boldsymbol{x},\boldsymbol{y},t,s) = \mathbb{E}\Big[\boldsymbol{\sigma}(\boldsymbol{x},t)\mathrm{d}\boldsymbol{B}_t\big(\boldsymbol{\sigma}(\boldsymbol{y},s)\mathrm{d}\boldsymbol{B}_s\big)^T\Big]$$
$$= \delta(t-s)\mathrm{d}t \int_{\Omega} \breve{\boldsymbol{\sigma}}(\boldsymbol{x},\boldsymbol{z},t)\breve{\boldsymbol{\sigma}}^T(\boldsymbol{y},\boldsymbol{z},s)\mathrm{d}\boldsymbol{z}$$

- Variance tensor (per unit of time) :

$$\boldsymbol{a}(\boldsymbol{x},t) = \frac{Q(\boldsymbol{x},t)}{\mathrm{d}t} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T(\boldsymbol{x},t)$$

- Turbulent Kinetic Energy (TKE) :

$$\mathrm{TKE} = \frac{1}{2}\frac{\mathrm{tr}(\boldsymbol{a})}{\mathrm{d}t} \qquad (m^2 \cdot s^{-2})$$

Stochastic Reynolds Transport Theorem (SRTT)

We assume that $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = 0$ in the following.

- Rate of change of a scalar process θ within a volume transported by the stochastic flow :

$$\mathrm{d}\int_{\mathcal{V}(t)} \theta(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} = \int_{\mathcal{V}(t)} (\mathrm{D}_t\theta + \theta\,\boldsymbol{\nabla}\cdot\boldsymbol{w}^\star)\mathrm{d}\boldsymbol{x}$$

- Stochastic transport operator :

$$\mathrm{D}_t\theta \triangleq \mathrm{d}_t\theta + (\underbrace{\boldsymbol{w} - \frac{1}{2}\,\boldsymbol{\nabla}\cdot\boldsymbol{a}}_{\boldsymbol{w}^\star})\cdot\boldsymbol{\nabla}\,\theta\mathrm{d}t + \underbrace{\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t\cdot\boldsymbol{\nabla}\,\theta}_{\text{multiplicative noise}} - \underbrace{\boldsymbol{\nabla}\cdot(\frac{\boldsymbol{a}}{2}\boldsymbol{\nabla}\theta)\mathrm{d}t}_{\text{subgrid diffusion}}$$

$\boldsymbol{w}^\star$: corrected drift – effect of statistical inhomogeneity of the small-scale flow component; generalization of the Stokes drift

[Bauer, Chandramouli, Chapron, Li & Mémin 2019a]

Conservation Laws

We assume that $\boldsymbol{\nabla}\cdot\boldsymbol{w}^\star = 0$ in the following.

- Conservation of extensive scalar :

$$\mathrm{D}_t\theta = 0$$

- Conservation of tracer energy : *[Resseguier, Memin & Chapron, 2017a]*

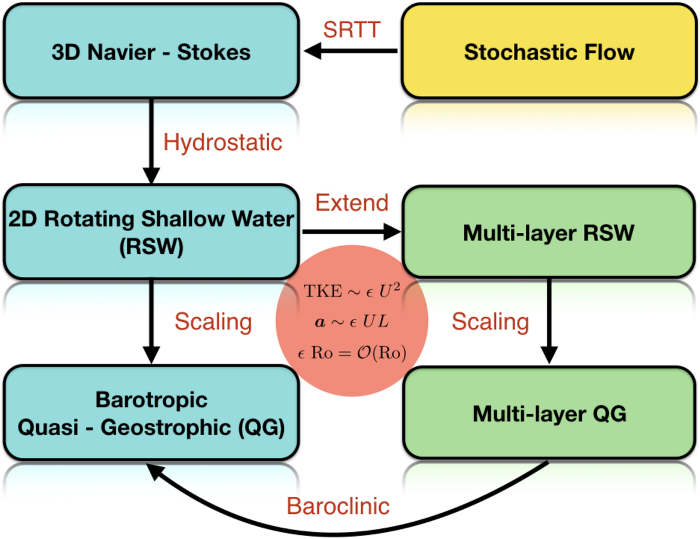
$$\begin{aligned}\mathrm{d}\int_\Omega \frac{1}{2}\theta^2 &= \int_\Omega \theta \mathrm{d}_t\theta + \frac{1}{2}\mathrm{d}\langle\theta\rangle_t \\ &= \int_\Omega \frac{1}{2}\theta^2\ \boldsymbol{\nabla}\cdot\left(\boldsymbol{w}^\star\mathrm{d}t + \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t\right) = 0\end{aligned}$$

- Energy decomposition of mean and variance fields :

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}(\mathbb{E}[\theta])^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\mathrm{Var}[\theta]$$

Outline

Stochastic Barotropic Vorticity Equation (SBVE)



Stochastic Barotropic Vorticity Equation (SBVE)

- Forced–dissipative advection of the potential vorticity (PV) q with source process :

$$\mathrm{D}_t q = ({\color{red}S_1} + F + D)\mathrm{d}t + {\color{red}S_2 \mathrm{d}\boldsymbol{B}_t}$$

$$S_1 = \frac{1}{2}\sum_{i,j=1,2} \partial^2_{ij}(\boldsymbol{\nabla}^\perp a_{ij} \cdot \boldsymbol{u}), \qquad S_2 \mathrm{d}\boldsymbol{B}_t = -\mathrm{tr}\Big(\boldsymbol{\nabla}^\perp(\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t)^T \boldsymbol{\nabla}\boldsymbol{u}^T\Big)$$

- Kinematic relationship between PV and stream function ψ :

$$q = \nabla^2\psi + f, \qquad \boldsymbol{u} = \boldsymbol{\nabla}^\perp\psi$$

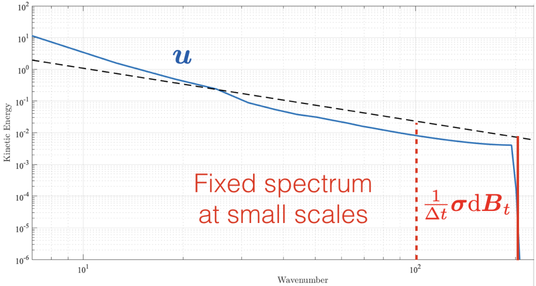
- Conservation of kinetic energy in the absence of D and F : *[Bauer, Chandramouli, Chapron, Li & Mémin 2019a]*

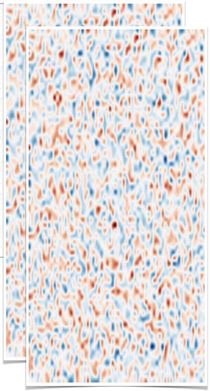
$${\color{red}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\|\boldsymbol{\nabla}\psi\|^2 = 0}$$

Outline

Parametrizations of Noise

- Homogeneous in space Stationary in time : *[Resseguier, Memin & Chapron, 2017b]*





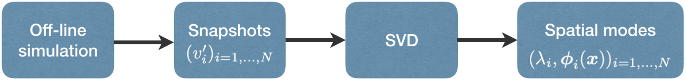
low-pass band filter $\star$ white noise

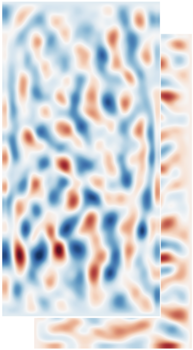
- Homogeneous Non-stationary : *[Resseguier, Li, Jouan, Derian, Mémin & Chapron, 2019]*
  Based on estimation of the absolute diffusivity spectral density (ADSD).

Parametrizations of Noise

- Heterogeneous Stationary : *[Chandramouli, Mémin, Chapron & Heitz 2019b]*

  Based on snapshot proper orthogonal decomposition (POD) method from data





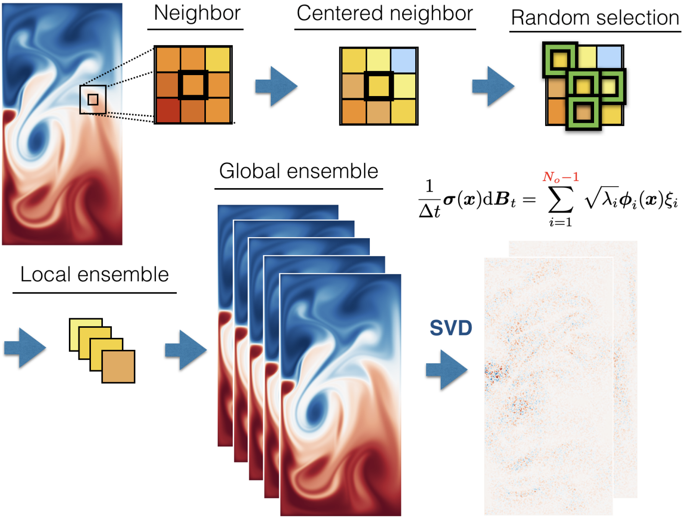
$$\boldsymbol{u}(\boldsymbol{x},t) = \bar{\boldsymbol{u}}(\boldsymbol{x}) + \sum_{i=1}^{n-1} \alpha_i(t)\boldsymbol{\phi}_i(\boldsymbol{x})$$

$$\frac{1}{\Delta t}\boldsymbol{\sigma}(\boldsymbol{x})\mathrm{d}\boldsymbol{B}_t = \sum_{i=n}^{N} \sqrt{\lambda_i}\boldsymbol{\phi}_i(\boldsymbol{x})\xi_i, \;\; \xi_i \sim \mathcal{N}(0,1)$$

$$\frac{1}{\Delta t}\boldsymbol{a}(\boldsymbol{x}) = \sum_{i=n}^{N} \lambda_i \boldsymbol{\phi}_i(\boldsymbol{x})\boldsymbol{\phi}_i^T(\boldsymbol{x})$$

Parametrizations of Noise

- Heterogeneous Non-stationary *[Li, Bauer, & Mémin 2019]* : On-line learning 'Pseudo-observations' from effective resolution



$$\frac{1}{\Delta t}\boldsymbol{\sigma}(\boldsymbol{x})\mathrm{d}\boldsymbol{B}_t = \sum_{i=1}^{N_o-1} \sqrt{\lambda_i}\boldsymbol{\phi}_i(\boldsymbol{x})\xi_i$$

Outline

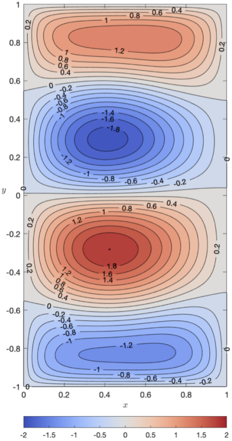
Long-term Diagnosis of Time–Statistics

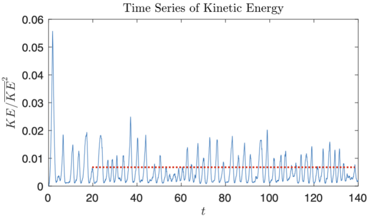
- Results predicted by DNS simulation :

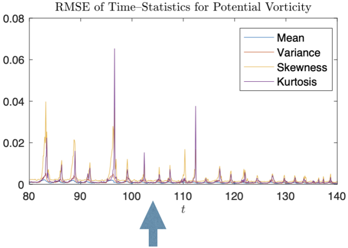
Instantaneous double-gyre circulation

highly variable

Time-Averaged four-gyre circulation

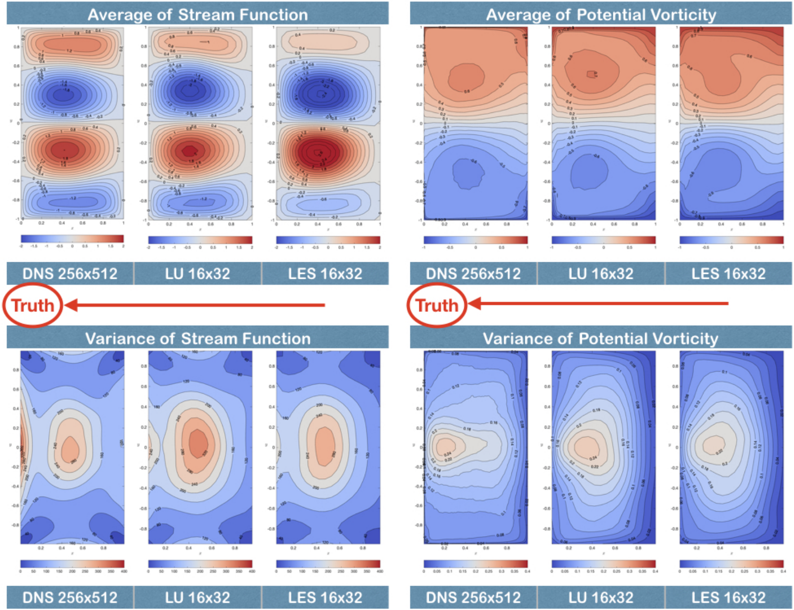




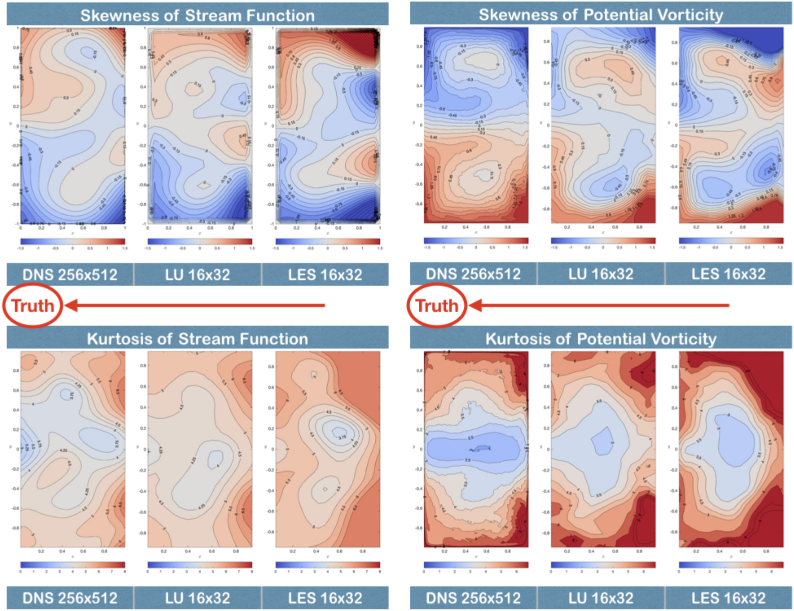


by perturbation of time-intervals

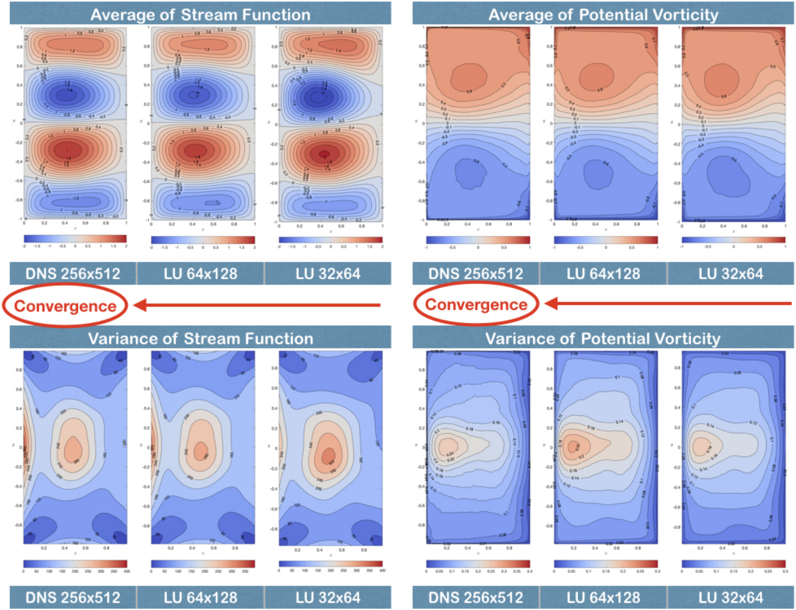
Long-term Diagnosis of Time–Statistics



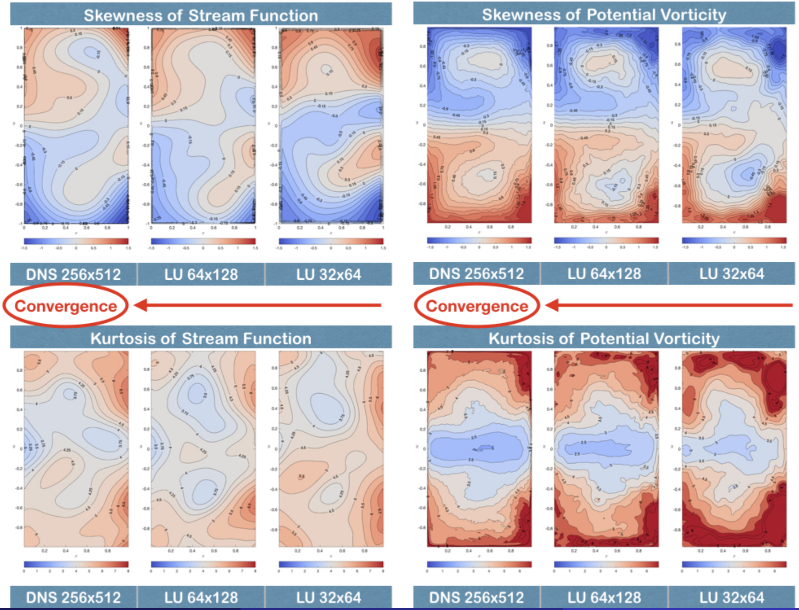
Long-term Diagnosis of Time–Statistics



Long-term Diagnosis of Time–Statistics



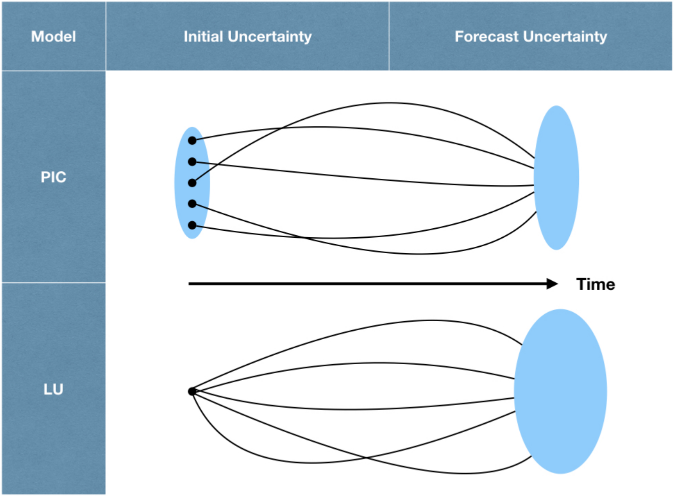
Long-term Diagnosis of Time–Statistics



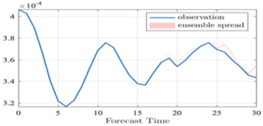
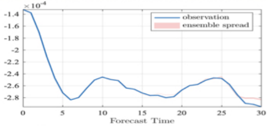
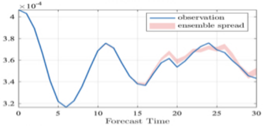
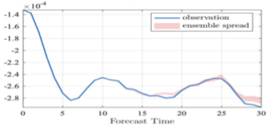
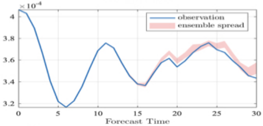
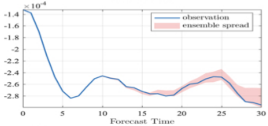
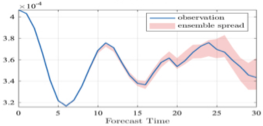
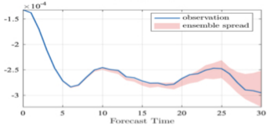
Outline

Short-term Verification of Ensemble Forecasts

Observed samples are the filtered DNS PV at each grid point; Ensemble simulations are performed with 30 particles from $t = 0$ to $t = 30$, starting from the (perturbed) filtered DNS.



Ensemble Spreads

Model	Patch1	Patch2
PIC homogeneous		
LU homogeneous		
PIC heterogeneous		
LU heterogeneous		

'better'

Reliability of Ensemble Forecasts

- Rank histogram

  (How well does the ensemble spread of the forecast represent the true uncertainty of the observations)

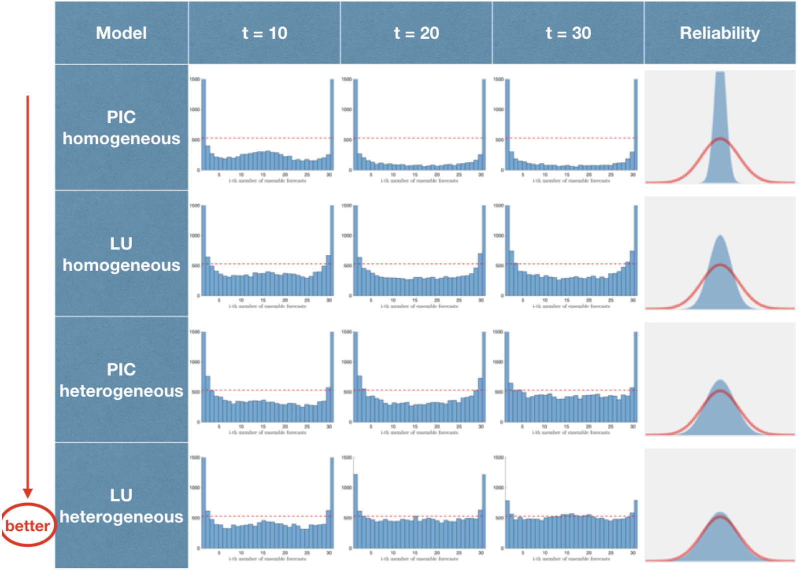
  Flat : ensemble represent well the observed probability distribution;

  U-shaped : ensemble spread too small, many observations falling outside the extremes of the ensemble;

  Dome-shaped : ensemble spread too large, most observations falling near the center of the ensemble;

  Asymmetric : ensemble contains bias.

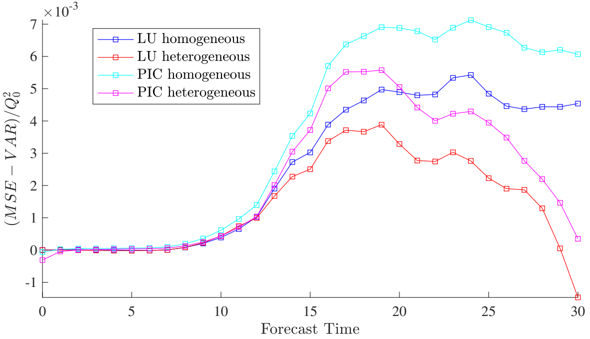
Reliability of Ensemble Forecasts



Reliability of Ensemble Forecasts

- The mean squared error (MSE) of the ensemble mean forecasts is identical to the average intra-ensemble sample variance (VAR), multiplied by an ensemble-size N – dependent inflation factor :

$$\overline{(q^o - \hat{\mathbb{E}}[q])^2}^{\boldsymbol{x}} = \frac{N+1}{N} \overline{\hat{\mathrm{Var}}[q]}^{\boldsymbol{x}}$$



- Other UQ metrics *[Resseguier, Li, Jouan, Derian, Mémin & Chapron, 2019]* : Continuous ranked probability score (CRPS), Energy score, Variogram score.

Conclusions

- Better eddy representation based on stochastic tranport;
- Capture better on a coarse mesh the correct long-term time-statistics;
- Spread is more accurate compared to PIC.

Work on ...

Strong interest in ensemble data assimilation with particle filter coupled with LU.

Thanks for Your Attention!